On integer solutions to $w^5 + x^5 = y^5 + z^n$ for $n = 3, 4, 5, 6$

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In this paper we consider Diophantine equations of the type $w^5 + x^5 = y^5 + z^n$, with $n =$ 3, 4, 5, 6. We derive an infinite set of solutions in Gaussian integers for the case $n = 5$.

I. INTRODUCTION

A famous open question [1] is the solution in positive integers of

$$w^5 + x^5 = y^5 + z^5. \tag{1}$$

While not settling this totally in integers, we give a solution with integers w and x where the right side y and z are Gaussian integers. The solutions to

$$w^n + x^n = y^n + z^n, \tag{2}$$

for $n = 4$ are well known to date back to Euler, [2] and the case where $n = 3$ is solved by the well known and celebrated "Taxicab numbers" named after the famous Hardy and Ramanujan anecdote. [3].
In this note we state some new integer cases for

$$w^5 + x^5 = y^5 + z^n, \tag{3}$$

for $n = 3$, 4 and 6.
Examples.

$$121^5 + 143^5 = 110^5 + 4114^3;$$

$$500^5 + 225^5 = 100^5 + 2375^4;$$

$$636^5 + 212^5 = 424^5 + 212^6.$$

II. GAUSSIAN INTEGER SOLUTIONS TO $a^5 + b^5 = c^5 + d^5$

We note here cases of (1) for integers w, x and Gaussian integers y, z.

$$\begin{aligned}
3^5 + 1^5 &= (2 + i3)^5 + (2 - i3)^5, \\
13^5 + 11^5 &= (12 + i17)^5 + (12 - i17)^5, \\
71^5 + 69^5 &= (70 + i99)^5 + (70 - i99)^5, \\
409^5 + 407^5 &= (408 + i577)^5 + (408 - i577)^5, \\
2379^5 + 2377^5 &= (2378 + i3363)^5 + (2378 - i3363)^5.
\end{aligned} \tag{4}$$

Anyone familiar with the classical Pell equation $1 + 2m^2 = n^2$, will recognize the well known continued fraction convergents featured in the right hand sides of these equations, leading to a fairly easy assertion that

Theorem 1 *An infinite number of solutions to* (1) *are given by*

$$(m+1)^5 + (m-1)^5 = (m+in)^5 + (m-in)^5, \tag{5}$$

where

$$m = \frac{(3+2\sqrt{2})^a - (3-2\sqrt{2})^a}{2\sqrt{2}}$$

and

$$n = \frac{(3+2\sqrt{2})^a + (3-2\sqrt{2})^a}{2} \tag{6}$$

for positive integers a.

A. Parametric solutions

Every solution of

$$w^5 + x^5 = y^5 + z^3 \tag{7}$$

generates a family of solutions

$$(wt^3)^5 + (xt^3)^5 = (yt^3)^5 + (zt^5)^3, \tag{8}$$

where $t = 1, 2, 3, ...$.
For instance,

$$121^5 + 143^5 = 110^5 + 4114^3;$$

generates

$$968^5 + 1144^5 = 8810^5 + 131648^3,$$
$$3267^5 + 3861^5 = 2970^5 + 999702^3,$$
$$...$$

Similarly, every solution of

$$w^5 + x^5 = y^5 + z^4 \tag{9}$$

generates a family of solutions

$$(wt^4)^5 + (xt^4)^5 = (yt^4)^5 + (zt^5)^4, \tag{10}$$

and solution of

$$w^5 + x^5 = y^5 + z^6 \tag{11}$$

generates a family of solutions

$$(wt^6)^5 + (xt^6)^5 = (yt^6)^5 + (zt^5)^6. \tag{12}$$

B. Another set of parametric solutions

$$w^5 + x^5 = y^5 + z^3$$

A parametric solution:

$$
\begin{aligned}
w &= m(m^5 + n^5 - p^5) \\
x &= n(m^5 + n^5 - p^5) \\
y &= p(m^5 + n^5 - p^5) \\
z &= (m^5 + n^5 - p^5)^2
\end{aligned} \tag{13}
$$

It's simultaneously a solution for

$$w^5 + x^5 = y^5 + z^6 :$$

w, x, y the same, $z = (m^5 + n^5 - p^5)$.
A parametric solution for

$$w^5 + x^5 = y^5 + z^4 :$$

$$
\begin{aligned}
w &= (m(m^{15} + n^{15} - p^{15}))^3 \\
x &= (n(m^{15} + n^{15} - p^{15}))^3 \\
y &= (p(m^{15} + n^{15} - p^{15}))^3 \\
z &= (m^{15} + n^{15} - p^{15})^4
\end{aligned} \tag{14}
$$

Our parametric solutions show the existence of infinite number of solutions to the equations (7, 9, 11).
Note that the parametric solutions (8, 10, 12) and (13 - 14) give only a small fraction of all solutions of (7, 9, 11).

III. CONCLUSIONS

In this work we studied Diophantine equations of the type $w^5 + x^5 = y^5 + z^n$, with $n = 3, 4, 6$. Examples of solutions, as well as parametric solutions were given.

Thanks who helped...

References:

[1] Find reference
[2] Dickson, L. E. History of the Theory of Numbers, Vol II, Ch XXII, page 644, originally published 1919 by Carnegie Inst of Washington, reprinted by The American Mathematical Society 1999.
[3] Hardy G. H. and Wright E.M., An Introduction to the Theory of Numbers, 3rd ed., Oxford University Press, London and NY, 1954, Thm. 412.
[4] Tito Piezas III ebook: https://sites.google.com/site/tpiezas/020 (fifth powers)